Correspondence Under Simplicial Toric Flops

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Abstract

In this paper, we discover a method to determine the equivalence correspondence of Chow motives under simplicial toric flops and carry it out on a typical example. The result saying that the graph closure still identifies the Chow motives on both sides is a first step for the study of Quantum invariance under singular toric flops.

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1 Introduction

In [3], authors showed that for a $\mathbf{P}^r$ flop $f : X \dashrightarrow X'$, the graph closure $[\overline{\Gamma}_f] \in A^*(X \times X')$ identifies the Chow motives $\hat{X}$ of X and $\hat{X}'$ of X'. Indeed, if $\mathscr{F} := [\overline{\Gamma}_f]$ then the transpose $\mathscr{F}^*$ is $[\overline{\Gamma}_{f^{-1}}]$. However, they also showed that for a Mukai flop $f : X \dashrightarrow X'$, the graph closure $[\overline{\Gamma}_f]$ must be added by some class $[\mathscr{E}]$ to be able to identify the Chow motives $\hat{X}$ of X and $\hat{X}'$ of X'. Indeed, the correspondence is given by

$$\mathscr{F} := [X \times_{\overline{X}} X'] = [\overline{\Gamma}_f] + [\mathscr{E}] \in A^*(X \times X').$$

In this research, we will devote ourselves to the study of toric case.

Let $X = X(\Delta)$ and $X' = X'(\Delta')$ which are connected by a flop. For the smooth 3-dimensional case, Danilov in late 70's proved that one can move the fan Δ to Δ' by a sequence of elementary flops. Geometrically each elementary flop is the blowing-up of a $(-1,-1)$ rational curve in a 3-fold then followed by a blowing-down of the exceptional divisor $\mathbf{P}^1 \times \mathbf{P}^1$ in another direction, that is, it is an easy ordinary flop.

M. Reid in early 80's generalized the above elementary move to higher dimensional case. Let $e_1, \ldots, e_{n+1}$ be primitive vectors in $N = \mathbb{Z}^n$ such that Δ_n and Δ_{n+1} be two top dimensional cones intersect along the face cone $W = \langle e_1, \cdots, e_{n-1}\rangle$, where $\Delta_j := \langle e_1, \cdots, \hat{e_j}, \cdots, e_{n+1}\rangle$. Let the linear relation between e_i's be

$$a_1 e_1 + \cdots + a_n e_n + e_{n+1} = 0.$$

Here we set $a_{n+1} = 1$ and we must have $a_n > 0$ since e_n and e_{n+1} lie in opposite sides of W. Reordering $e_1, \ldots, e_{n-1}$ we may assume that $a_i < 0$ for $1 \le i \le \alpha$, $a_i = 0$ for $\alpha + 1 \le i \le \beta$ and $a_i > 0$ for $\beta + 1 \le i \le n+1$. Notice that $0 \le \alpha \le \beta \le n-1$. The so-called "elementary move" is that our original two cones comes from the following two decompositions: when $\alpha \ge 2$, $\Delta := \langle e_1, \cdots, e_{n+1}\rangle$:

$$\Delta = \bigcup_{\beta+1 \le j \le n+1} \Delta_j = \bigcup_{1 \le j \le \alpha} \Delta_j.$$

We have the fact in hand that *any smooth toric flop is ordinary.* The proof was suggested by Hui-Wen Lin. Indeed, with the same situation as above, the smoothness condition tells us that the primitive generators $e_1, \ldots, e_n$ form a $\mathbb{Z}$-basis of the lattice N and so do $e_1, \ldots, e_{n-1}, e_{n+1}$. When we represent e_{n+1} as a $\mathbb{Z}$-linear combination of $e_1, \ldots, e_n$ and e_n as a $\mathbb{Z}$-linear combination of $e_1, \ldots, e_{n-1}, e_{n+1}$ simultaneously, we can get that $a_i = -1$ for $i = 1, \ldots, \alpha$ and $a_i = 1$ for $i = \beta + 1, \ldots, n$. For the flop case, all $e_1, \ldots, e_n, e_{n+1}$ should all lie in an affine hyperplane of $N_{\mathbb{Q}}$. This implies that

$$-\sum_{i=1}^{\alpha} a_i = \sum_{i=\beta+1}^{n} a_i + 1$$

and thus $\alpha = n + 1 - \beta$. Translating these data to the Reid's diagram, we have that $Z \to W$ is a bundle with fiber a projective space $\mathbb{P}^{n-\beta}$ and $Z' \to W$ is a bundle with fiber a projective space $\mathbb{P}^{\alpha-1}$. Note that $n - \beta = \alpha - 1$. It illustrates that the whole diagram for this case is an ordinary $\mathbb{P}^{\alpha-1}$-flop. Hence we conclude that *for smooth toric flops, the graph closure identifies the two Chow motives and thus the interesting problem is about singular toric flops.*

The main result of this paper is stated as follows. By modifying the pair of fans (F, F') corresponding to a simple $\mathbf{P}^r$-flop, let $v_0, \ldots, v_r, w_0, \ldots, w_{s-1}$ be primitive vectors in $\mathbf{N}$ which form a $\mathbf{Q}$-basis and w_s in N such that for some $x_i, y_j \in N$ for $0 \le i \le r$, $0 \le j \le s$, with $\gcd(x_0, \ldots, y_s) = 1$,

$$x_0 + \cdots + x_r = y_0 + \cdots + y_s$$

$$x_0 v_0 + \cdots + x_r v_r = y_0 w_0 + \cdots + y_s w_s.$$

To compactify the corresponding toric varieties, we add a vector $v_{r+1} \in N$ to the fans, say $v_{r+1} = -x_0 v_0 - \cdots - x_r v_r$. The resulting varieties with at most quotient singularities will be $X := X(F)$, $X' := X(F')$, which are connected by a flop.

Let L be sublattice of N generated by $v_0, \ldots, w_{s-1}$.

Theorem A. (=Corollary 3.1) *If $L \subset N$ has index ℓ, then the torsion elements of $A_*(X)$ have order dividing ℓ.*

As a lemma, we show that when $L = N$, $A_*(X)$ can be shown to be free and as a corollary, when we consider the case of $x_i, y_j = 1$ $\forall i, j$, it is useful for constructing torsions.

Theorem B. (=*Theorem 3.1*) *If* $L = N,\ r = s,\ x_r = y_r = 1,\ x_{i+1}|x_i,\ y_{j+1}|y_j$, *then there exists a correspondence* $\mathscr{F} \in A_*(X \times X')$ *inducing an isomorphism of "integral" Chow groups* $A_*(X) \to A_*(X')$ *with inverse* $\mathscr{F}^* \in A_*(X' \times X)$. *In fact, the correspondence is exactly the graph closure.*

Also, we can get the more general result as follows.

Theorem C. (=*Theorem 3.2*) *If* $r = s, x_r = y_r = 1$, *then the graph closure* $[\overline{\Gamma}_f] \in A_*(X \times X')$ *induces an isomorphism between "rational" Chow groups* $A_*(X)_{\mathbf{Q}} \to A_*(X')_{\mathbf{Q}}$, *the inverse is* $[\overline{\Gamma}_{f^{-1}}] \in A_*(X' \times X)$.

As a warm-up, in Section 2, we give another proof for the partial result in ([3]) that for a simple $\mathbf{P}^r$-flop, the graph correspondence will identify the two Chow motives. In Section 3, we enter the proof of our main result. In Section 4, we give two examples : the 1st one contains an extra proposition which provides us the evidence to start the next task and the 2nd one contains the stronger result for Theorem C saying that the graph closure induces an isomorphism between "integral" Chow groups.

2 $[\overline{\Gamma}_f]$ as the correspondence under simple $\mathbf{P}^r$-flops

2.1 Decomposition of $[\overline{\Gamma}_f]$ in $A^*(X \times X')$

In [2], Fulton and Sturmfels provided a way to decompose the closure of subtorus in X into linear combination of toric strata in $A^*(X \times X')$. It will benefit us to decompose the graph closure $\overline{\Gamma}_f$ in $X \times X'$. For stating the decomposition theorem, we recall some terminologies used in [FS]. Fixing a saturated d-sublattice L of N, let T_L be the corresponding subtorus of T_N. For any $v \in N$, let

$$\Delta(v) = \{\sigma \in \Delta : (L_{\mathbf{R}} + v) \cap \sigma = \{pt\}\}.$$

We say v is *generic* for $L_{\mathbf{R}}$ if $dim(\sigma) = n - d\ \forall \sigma \in \Delta(v)$ (in general $\dim(\sigma) \le n - d$).

Fan displacement rule. (= Theorem 3.2 in [2]) *Let* Y *be the closure of* T_L *in* X, *if* v *is a generic lattice point then*

$$[Y] = \sum_{\sigma \in \Delta(v)} m_\sigma \cdot [V(\sigma)] \text{ in } A_d(X) \text{ where } m_\sigma = [N : L + N_\sigma].$$

By this rule, decomposing $[\overline{\Gamma}_f] = [\overline{T_{N \times_N N}}] \in A(X \times X')$ is equivalent to finding an element $v \in N$ such that $\forall \sigma \in \Delta,\ \sigma' \in \Delta'$,

$$(\sigma + v) \cap \sigma' = \{pt\} \text{ implies } \dim \sigma + \dim \sigma' = n.$$

More precisely, if $(v, 0)$ is generic for the sublattice $N \times_N N$,

$$[\overline{\Gamma}_f] = \sum_{\sigma \times \sigma' \in \Delta((v,0))} m_{\sigma \times \sigma'} [V(\sigma) \otimes V(\sigma')] \in A^*(X \times X') \simeq A^*(X) \otimes A^*(X')$$

where $m_{\sigma \times \sigma'} = [N \times N : N \times_N N + N_\sigma + N_{\sigma'}] = [N : N_\sigma + N_{\sigma'}]$, the last equality can be seen by mapping $N \times N$ onto N via $(a, b) \mapsto a - b$.

2.2 Another proof

Let $X = \mathbf{P}_{\mathbf{P}^r}(\mathscr{O}(-1)^{\oplus r+1} \oplus \mathscr{O})$ which, as a toric variety, corresponds to the complete fan Δ with $\Delta^{(n)}$ consisting of $\langle v_0, \dots, \hat{v}_i, \dots, v_{r+1}, w_0, \dots, \hat{w}_j, \dots, w_r \rangle_+$, $0 \le i \le r+1$, $0 \le j \le r$ where v_i, w_j satisfy the (only) relation

$$v_0 + \cdots + v_r = w_0 + \cdots + w_r = -v_{r+1}.$$

Then X' corresponds to $\Delta'^{(n)}$ consisting of $\langle v_0, \dots, \hat{v}_i, \dots, v_r, w_0, \dots, \hat{w}_j, \dots, w_{r+1} \rangle_+$, $0 \le i \le r$, $0 \le j \le r+1$, here $w_{r+1} := v_{r+1}$. Notice that $X' \cong \mathbf{P}_{\mathbf{P}^r}(\mathscr{O}(-1)^{\oplus r+1} \oplus \mathscr{O})$ and (f, X, X') is so-called a simple $\mathbf{P}^r$-flop.

Proposition 2.1. *$v = v_0 + 2v_1 + \cdots + (r+1)v_r + (r+2)w_0 + \cdots + (2r+1)w_{r-1}$ is an element such that $(v,0)$ is generic for the sublattice $N \times_N N$.*

A trivial but crucial observation: If $(\sigma + v) \cap \tau = \{pt\}$, then $\sigma \cap \tau = \{0\}$. Denote $\{0, \dots, r\}$ by $[r]$. The cones in Δ are of the form $\sigma_{IJ} = \langle v_i, w_j : i \in I, j \in J \rangle_+$ with $I \subsetneq [r+1]$, $J \subsetneq [r]$ and in Δ' are of the form $\sigma_{I'J'} = \langle v_i, w_j : i \in I', j \in J' \rangle_+$ with $I' \subsetneq [r]$, $J' \subsetneq [r+1]$. An element in σ_{IJ} will be denoted by $\sum a_i v_i + \sum b_j w_j$. An element $x \in (\sigma_{IJ} + v) \cap \sigma_{I'J'}$ corresponds to a solution $\{(a_i, b_j)_{i \in I, j \in J}, (a'_i, b'_j)_{i \in I', j \in J'}\}$ in $\mathbf{R}^{\ge 0} \times \mathbf{R}^{\ge 0}$ satisfying

$$\sum_{i \in I} a_i v_i + \sum_{j \in J} b_j w_j + v = \sum_{i \in I'} a'_i v_i + \sum_{j \in J'} b'_j w_j. \tag{1}$$

Thus, $(\sigma_{IJ}, \sigma_{I'J'}) \in \Delta(v) (:= \Delta((v,0)))$ implies $I \cap I' = J \cap J' = \emptyset$ and $r+1 \notin I \cap J'$.

Proof of proposition. Suppose $(\sigma_{IJ} + v) \cap \sigma_{I'J'} = \{pt\}$ and $\{v_0, \dots, v_r, w_0, \dots, w_{r-1}\}$ is a $\mathbf{Z}$-basis for $\mathbf{N}$.

Case 1. $r+1 \notin I$, $r \in J$.
By the above observation, $r \notin J'$. For $i (\neq r+1) \in I$, the coefficient of v_i in LHS is equal to $a_i + (i+1) + b_r > 0$, and is equal to 0 in RHS ($r \notin J'$), so there is no solution and thus $I = \emptyset$. But this implies $i + 1 + b_r = 0$, a contradiction, so this case is omitted.

Case 2. $r+1 \in I$, $r \notin J$. ($\Rightarrow r+1 \notin J'$)
For $j \in J$, the coefficient of w_j in LHS is equal to $b_j + r + 2 + j > 0$ but in RHS is ≤ 0 since $j \notin J'$ and w_r has negative contibution along w_j for $j \in [r-1]$, so $J = \emptyset$. Then LHS has component $r+2+j$ along w_j, $j \in [r-1]$ and thus $[r-1] \subset J'$ otherwise RHS is nonpositive along w_j.
For $i \in [r] \setminus (I \cup I')$, $-a_{r+1} + i + 1 = b'_r$, so $[r] \setminus (I \cup I') = \emptyset$ or $[r] \setminus (I \cup I') = \{i^*\}$.

Case 2.1 $J' = [r]$.
For $i \in I \setminus \{r+1\}$, $a_i - a_{r+1} + i + 1 = b'_r$, while for $i \in I'$, $-a_{r+1} + i + 1 = a'_i + b'_r$, combining these together,

$$i + 1 \le a_{r+1} + b'_r \le i' + 1 \; \forall i \in I \setminus \{r+1\},\ i' \in I',$$

in particular $i < i' \; \forall i \in I \setminus \{r+1\}$, $i' \in I'$. Let

$$i_{max} = \begin{cases} \max I \setminus \{r+1\}, & I \setminus \{r+1\} \neq \\ -1, & else \end{cases}, \quad i'_{min} = \begin{cases} \min I', & I' \neq \emptyset \\ r+1, & else \end{cases}.$$

Case 2.1.1 $[r] \setminus (I \cup I') = \emptyset$.
In this case, the equation has solutions

$$a_i = i_{max} - i \; \forall i \in I \setminus \{r+1\},\; a'_i = i - i_{max} \; \forall i \in I',\; a_{r+1} + b'_r = i_{max} + 1$$

and

$$a_i = i'_{min} - i \; \forall i \in I \setminus \{r+1\},\; a'_i = i - i_{min} \; \forall i \in I',\; a_{r+1} + b'_r = i_{min} + 1$$

so this is not the case for uniqueness.

Case 2.1.2 $[r] \setminus (I \cup I') = \{i^*\}$.
In this case, the equation has solutions

$$a_i = i^* - i \; \forall i \in I \setminus \{r+1\},\; a'_i = i - i^* \; \forall i \in I',\; a_{r+1} + b'_r = i^* + 1$$

and $a_{r+1} + b'_r = i^* + 1$ has more than one solution, so this is not the case also.

Case 2.2 $J' = [r-1]$.
For $i \in I \setminus \{r+1\},\; a_i - a_{r+1} + i + 1 = 0$, while for $i \in I',\; -a_{r+1} + i + 1 = a'_i$, combining these together, we get

$$i + 1 \leq a_{r+1} \leq i' + 1 \; \forall i \in I \setminus \{r+1\},\; i' \in I'.$$

By the same analysis as in previous case, the case of $[r] \setminus (I \cup I') = \emptyset$ is omitted, but for the case of $[r] \setminus (I \cup I') = \{i^*\}$, $I \setminus \{r+1\} < i^* < I'$, the only solution is

$$a_{r+1} = i^* + 1,\; a_i = i^* - i \; \forall i \in I \setminus \{r+1\},\; a'_i = i - i^* \; \forall i \in I' \; b'_j = r + 2 + j \; \forall j \in J.$$

Hence in this case

$$(\star)\; I = \{0, \ldots, i^* - 1\} \cup \{r+1\},\; I' = \{i^* + 1 \ldots, r\},\; J = \emptyset,\; J' = \{0, \ldots, r-1\}$$

for some $0 \leq i^* \leq r$.

Case 3. $r+1 \in I \; r \in J$. ($\Rightarrow r, r+1 \notin J'$).
For $i \in [r] \setminus (I \cup I'),\; i + 1 + b_r - b_{r+1} = 0$, so either $[r] \setminus (I \cup I') = \emptyset$ or $[r] \setminus (I \cup I') = \{i^*\}$.
For $j \in [r-1] \setminus (J \cup J'),\; -b_r + r + 2 + j = 0$, so either $[r-1] \setminus (J \cup J') = \emptyset$ or $[r-1] \setminus (J \cup J') = \{j^*\}$. For the existance and uniqueness reason as in the previous case, in this case

$$(\star\star)\; I = \{0, \ldots, i^* - 1\} \cup \{r+1\},\; I' = \{i^* + 1 \ldots, r\},$$
$$J = \{0, \ldots, j^* - 1\} \cup \{r\},\; J' = \{j^* + 1, \ldots, r-1\}$$

for some $0 \leq i^* \leq r,\; 0 \leq j^* \leq r - 1$.

Case 4. $r+1 \notin I,\; r \notin J$.
For $j \in [r-1] \setminus (J \cup J')$, the coefficient of w_j in LHS is equal to $r + 2 + j > 0$ but in RHS≤ 0, so $[r-1] \setminus J \cup J' = \emptyset$.
For $j \in J$, the coefficient of w_j in LHS is equal to $b_j + r + 2 + j > 0$ but in RHS ≤ 0, so $J = \emptyset$ and thus $[r-1] \subset J'$.

Case 4.1. $r \notin J'$.

For $i \in [r] \setminus I \cup I'$, the coefficient of v_i in LHS is equal to $i+1 > 0$ but in RHS ≤ 0, so $[r] \subset I \cup I'$.

For $i \in I$, the coefficient of v_i in LHS is equal to $a_i + i + 1 > 0$ but in RHS ≤ 0, so $I = \emptyset$ and $[r] \subset I'$ but $I' \subsetneq [r]$. $\rightarrow\leftarrow$.

Case 4.2. $J' = [r]$.

For $i \in [r] \setminus I \cup I', i+1 = b'_r$, so either $[r] \setminus (I \cup I') = \emptyset$ or $[r] \setminus (I \cup I') = \{i^*\}$. For $i \in I$, $a_i + i + 1 = b'_r$ while for $i \in I'$, $i+1 = a'_i + b'_r$, in particular, $i+1 \leq b_r \leq i'+1 \ \forall i \in I, i' \in I'$.

For $j \in J' \setminus \{r+1\}$, $b'_j = r+2+j$.

If $[r] \setminus (I \cup I') = \emptyset$, then $b'_r = i_{max} + 1$ and $b'_r = i'_{min} + 1$ which generate distinct solutions from the previous case. But for $[r] \setminus (I \cup I') = \{i^*\}$, $I < i^* < I'$, the unique solution is

$$b'_r = i^* + 1,\ a_i = i^* - i \ \forall i \in I,\ a'_i = i - i^* \ \forall i \in I' m\ b'_j = r+2+j \ \forall j \in J'.$$

So in this case

$$(\star\star\star) I = \{0, \ldots, i^* - 1\},\ I' = \{i^* + 1, \ldots, r\},\ J = \emptyset,\ J' = \{0, \ldots, r\}$$

for some $0 \leq i^* \leq r$.

Since $J' \subsetneq [r+1]$, all possible cases are exhausted. For (I, J), (I', J') satisfying $(\star)$ or $(\star\star)$ or $(\star\star\star)$, $\dim \sigma_{IJ} + \dim \sigma_{I'J'} = |I| + |J| + |I'| + |J'| = 2r+1 = n$. Hence v is generic and $\Delta(v) = (\star) \cup (\star\star) \cup (\star\star\star)$.

□

Since X and X' are nonsingular, $V(\sigma_{IJ}) = \prod_{i \in I} D_{v_i} \cdot \prod_{j \in J} D_{w_j}\ V(\sigma_{I'J'}) = \prod_{i \in I'} D_{v_i} \cdot \prod_{j \in J'} D_{w_j}$. In terms of $h,\ \xi,\ h',\ \xi'$, in the three cases $(\star)$, $(\star\star)$, $(\star\star\star)$,

$$\begin{array}{lll} V(\sigma_{IJ}) = (\xi - h)^i \xi & V(\sigma_{I'J'}) = h'^{r-i}(\xi' - h')^r & (0 \leq i \leq r) \\ V(\sigma_{IJ}) = (\xi - h)^i \xi h^{j+1} & V(\sigma_{I'J'}) = h'^{r-i}(\xi' - h')^{r-1-j} & (0 \leq i \leq r,\ 0 \leq j \leq r-1) \\ V(\sigma_{IJ}) = (\xi - h)^i & V(\sigma_{I'J'}) = h'^{r-i}(\xi' - h')^{r+1} & (0 \leq i \leq r) \end{array}$$

respectively and in any case $m_{\sigma_{IJ} \times \sigma_{I'J'}} = 1$. Hence

$$[\overline{\Gamma}_f] = \sum_{i=0}^{r} \sum_{j=0}^{r} \xi (\xi - h)^i h^j \otimes h'^{r-i} (\xi' - h')^{r-j} + \sum_{i=0}^{r} (\xi - h)^i \otimes h'^{r-i} (\xi' - h')^{r+1}.$$

Let $\mathscr{F}_0 = \sum_{i=0}^{r} \sum_{j=0}^{r} \xi (\xi - h)^i h^j \otimes h'^{r-i} (\xi' - h')^{r-j}$, $\mathscr{F}_1 = \sum_{i=0}^{r} (\xi - h)^i \otimes h'^{r-i} (\xi' - h')^{r+1}$. From the relation $h^{r+1} = 0$,

$$\mathscr{F}_0 = \sum_{i=0}^{r} \sum_{j=0}^{r} ((\xi - h)^{i+1} h^j + (\xi - h)^i h^{j+1}) \otimes h'^{r-i} (\xi' - h')^{r-j}.$$

Corollary 2.1. $[\overline{\Gamma}_f] = \mathscr{F}_0 + \mathscr{F}_1$ *where*

$$\mathscr{F}_0 = \sum_{i=0}^{r}\sum_{j=0}^{r}((\xi-h)^{i+1}h^j + (\xi-h)^i h^{j+1}) \otimes h'^{r-i}(\xi'-h')^{r-j}$$

and

$$\mathscr{F}_1 = \sum_{i=0}^{r}(\xi-h)^i \otimes h'^{r-i}(\xi'-h')^{r+1}.$$

Proposition 2.2. *For* $0 \le p \le r+1,\ 0 \le q \le r,\ \mathscr{F}_0((\xi-h)^p h^q) = h'^p(\xi'-h')^q$.

Proof. We fix the canonical **Z**-basis $\{(\xi-h)^p h^q\ ,0 \le p \le r+1,\ 0 \le q \le r\}$ for $A_*(X)$. Let $p_1 : X \times X' \to X,\ p_2 : X \times X' \to X'$ be the projection maps. Clearly $p_1^*((\xi-h)^p h^q) = (\xi-h)^p h^q \otimes 1$. Recall the degree map of $A_0(X) = \mathbf{Z} \cdot pt$ by sending $k \cdot pt$ to k. We have that $(p_2)_*([V \otimes W]) = \begin{cases} \deg(V)[W], & [V] \in A_0(X) \\ 0, & [V] \in A_{\ge 1}(X) \end{cases}$.

$$\begin{aligned}\mathscr{F}_0((\xi-h)^p h^q) &= (p_2)_*(\sum_{i=0}^{r}\sum_{j=0}^{r}((\xi-h)^{i+p+1}h^{j+q} + (\xi-h)^{i+p}h^{j+q+1}) \otimes h'^{r-i}(\xi'-h')^{r-j}) \\ &= \sum(\deg(\xi-h)^{i+p+1}h^{j+q} + \deg(\xi-h)^{i+p}h^{j+q+1})h'^{r-i}(\xi'-h')^{r-j}\end{aligned}$$

where the sum is over $0 \le i \le r,\ 0 \le j \le r,\ i+j+p+q = 2r$. Note that for $i \ge 0,\ i+j = 2r+1$,

$$\deg(\xi-h)^i h^j = \begin{cases} (-1)^{r-j}, & j \le r \\ 0, & j \ge r+1 \end{cases}$$

which can be proved by induction via the fact that $h^{r+1} = 0,\ \xi(\xi-h)^{r+1} = 0$. Hence

$$\begin{aligned}\mathscr{F}_0((\xi-h)^p h^q) = & \sum_{0\le i\le r,\ 0\le j\le r,\ i+j+p+q=2r,\ j+q\le r}(-1)^{r-j-q}h'^{r-i}(\xi'-h')^{r-j} \\ & + \sum_{0\le i\le r,\ 0\le j\le r,\ i+j+p+q=2r,\ j+q\le r-1}(-1)^{r-j-q-1}h'^{r-i}(\xi'-h')^{r-j} \\ & = \sum_{0\le i\le r,\ 0\le j\le r,\ i+j+p+q=2r,\ j+q=r}(-1)^{r-j-q}h'^{r-i}(\xi'-h')^{r-j} \\ & + \sum_{0\le i\le r,\ 0\le j\le r,\ i+j+p+q=2r,\ j+q\le r-1}((-1)^{r-j-q} + (-1)^{r-j-q-1})h'^{r-i}(\xi'-h')^{r-j} \\ & = h'^p(\xi'-h')^q.\end{aligned}$$

□

Proposition 2.3. $\mathscr{F}_1((\xi-h)^p h^q) = \begin{cases} (-1)^{r-q}h'^{p+q-r-1}(\xi'-h')^{r+1}, & p+q \ge r+1 \\ 0, & p+q \le r. \end{cases}$

Proof.

$$\begin{aligned}\mathscr{F}_1((\xi-h)^p h^q) &= (p_2)_*(\sum_{i=0}^{r}(\xi-h)^{i+p}h^q\otimes h'^{r-i}(\xi'-h')^{r+1})\\ &= \sum_{0\le i\le r,\ i+p+q=2r+1}(-1)^{r-q}h'^{r-i}(\xi'-h')^{r+1}\\ &= \begin{cases}(-1)^{r-q}h'^{p+q-r-1}(\xi'-h')^{r+1}, & p+q\ge r+1\\ 0, & \textit{else}.\end{cases}\end{aligned}$$

□

Corollary 2.2. *$\mathscr{F} := [\overline{\Gamma}_f] \in A^*(X\times X')$ induces an isomorphism $\hat{X} \xrightarrow{\sim} \hat{X}'$ with the inverse $\mathscr{F}^*$ defined by symmetry. In fact, it induces a group isomorphism $\mathscr{F} : A_*(X) \to A_*(X')$,*

$$(\xi-h)^p h^q \mapsto \begin{cases}h'^p(\xi'-h')^q + (-1)^{r-q}h'^{p+q-r-1}(\xi'-h')^{r+1}, & p+q\ge r+1\\ h'^p(\xi'-h')^q, & p+q\le r.\end{cases}$$

Proof. Since $\mathscr{F} = \mathscr{F}_0 + \mathscr{F}_1$, the mapping follows from the above propositions. One can directly show that $\mathscr{F} : A_*(X) \to A_*(X')$ has the inverse $\mathscr{F}^* : A_*(X') \to A_*(X)$. By Manin's identity principle, showing that $\mathscr{F} : \hat{X} \to \hat{X}'$ has the inverse $\mathscr{F}^*$ is equivalent to showing that $\mathscr{F}_T : A_*(T\times X) \to A_*(T\times X')$ has the inverse $\mathscr{F}_T^*$ for all toric T, but since $A_*(T\times X) \simeq A_*(T)\otimes A_*(X)$, for X,T toric, one can show that $\mathscr{F}_T = id_{A_*(T)} \otimes \mathscr{F}|_{A_*(X)\to A_*(X')}$ and thus an isomorphism with the inverse $id_{A_*(T)} \otimes \mathscr{F}^*|_{A_*(X')\to A_*(X)} = (\mathscr{F}^*)_T$. □

If $H \in A_*(X)$ is the hyperplane class of the exceptional divisor $\mathbf{P}^r \simeq V(\sigma_{[r],\emptyset})$, then $H = V(\sigma_{[r],\{0\}})$ and $H^\ell = V(\sigma_{[r],[\ell-1]}) = (\xi-h)^{r+1}h^\ell$. Hence

$$\mathscr{F}(H^\ell) = (-1)^{r-\ell}h'^\ell(\xi'-h')^{r+1} = (-1)^{r-\ell}H'^\ell.$$

3 Main results

3.1 The result for general (L,N)

First of all, I recall that the graph closure $[\overline{\Gamma}_f] = [\overline{T_{N\times_N N}}] \in A(X\times X')$. In the special singular case defined in the introduction, I will show that $[\overline{T_{N\times_N N}}]$ does not necessarily induce an isomorphism of chow motives (over $\mathbf{Q}$). Denoted by $L \subset N$ the sublattice generated by the $\mathbf{Q}$-basis $v_0,\dots,v_r,w_0,\dots,w_{s-1}$. The two distinct simplicial subdivisions of $\langle v_0,\dots,v_r,w_0,\dots w_s\rangle$ define a flop (f,F,F') as in [5]. We add to the fans a vector $v_{r+1}\in N$ such that we can compactify them into a proper toric varieties, here, $v_{r+1} := -x_0v_0 - \cdots - x_rv_r$. More explicitly, F has top cones

$$\sigma^{ij} = \langle v_0,\dots,\hat{v}_i,\dots,v_{r+1},w_0,\dots,\hat{w}_j,\dots,w_s\rangle_+,\ 0\le i\le r+1,\ 0\le j\le s$$

and F' has top cones

$$\sigma^{ij} = \langle v_0,\dots,\hat{v}_i,\dots,v_r,w_0,\dots,\hat{w}_j,\dots,w_{s+1}\rangle_+,\ 0\le i\le r,\ 0\le j\in s+1$$

with $w_{s+1} := v_{r+1}$. For simplicity, I denote $\{i \in \mathbf{Z} : a \le i \le b\}$ by $[a,b]$, $[b] := [0,b]$ Denote the corresponding toric varieties by $X(F,N)$ and $X(F',N)$ respectively with ambient lattice N being emphasised.

It is well-known that the rational chow ring $A^*(X)_{\mathbf{Q}}$ is isomorphic to $\mathbf{Q}[\xi,h]/(h^{r+1},(\xi-h)^{r+1}\xi)$, where ξ and h correspond to $[D_{v_{r+1}}]$ and $(1/y_r)[D_{w_r}]$ respectively. Under this identification,

$$D_{w_i} = y_i \cdot h \,\forall 0 \le i \le r-1,\ D_{v_i} = x_i(\xi - h)\ \forall 0 \le i \le r,\ D_{r+1} = \xi.$$

Lemma 3.1. *If $L = N$, then the chow group $A_*(X(F,N))$ is free, it is isomorphic to $H_{2*}(X(F,N))$ and it has $\mathbf{Z}$-basis*

$$\{V(\sigma_{[i,r+1],[j,s]}) : 1 \le i \le r+2,\ 1 \le j \le s+1\}.$$

Proof. Observe that F has $(r+2)(s+1)$ top cones. For each, $0 \le k \le (r+2)(s+1)-1$, there exists unique pair $(p_k,q_k) \in [s]\times[r+1]$ such that $k = p_k(r+2)+q_k$, and we define $\sigma_k = \sigma^{r+1-q,s-p}$. I claim that

$$\sigma_0,\dots,\sigma_{(r+2)(s+1)-1}$$

is a good ordering such that it gives X a cellular decomposition.

For $i \in [(r+2)(s+1)-1]$, let $\tau_i = \sigma_i \cap \{\sigma_k : k > i, \sigma_k \cap \sigma_i \text{ is a facet of } \sigma_i\}$, and for $k > i$,

$$\sigma_k \cap \sigma_i = \begin{cases} \sigma_{[r+1]\setminus\{r+1-q_k,r+1-q_i\},[s]\setminus\{s-p_i\}}, & p_k = p_i,\ q_k > q_i \\ \sigma_{[r+1]\setminus\{r+1-q_i\},[s]\setminus\{s-p_k,s-p_i\}}, & p_k > p_i, q_k = q_i \\ \text{codim} \ge 2 \text{ in } \sigma_i, & \text{else.} \end{cases}.$$

Then

$$\tau_i = \sigma_{[r+1]\setminus[r+1-q_i],[s]\setminus s-p_i} \cap \sigma_{[r+1]\setminus r+1-q_i,[s]\setminus[s-p_i]} = \sigma_{[r+1]\setminus[r+1-q_i],[s]\setminus[s-p_i]}$$

and $\tau_i < \sigma_k$ implies $r+1-q_k \in [r+1-q_i]$, $s-p_k \in [s-p_i]$, i.e. $q_k \ge q_i$, $p_k \ge p_i$ which implies $k \ge i$.

Let $Y_i = V(\tau_i) \cap U_{\sigma_i}$. By definition, it is the affine toric variety in N/N_τ defined by $\langle v_0,\dots,v_{r-q_i},w_0,\dots,w_{s-p_i-1}\rangle$, whose generators form a $\mathbf{Z}$-basis of N/N_τ. Hence Y_i is smooth and isomorphic to $\mathbf{C}^{r+s+1-p_i-q_i}$. For $0 \le i \le (r+2)(s+1)-1$, if we define $Z_i = Y_i \cup \cdots \cup Y_{(r+2)(s+1)-1}$, then

$$X = Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_{(r+2)(s+1)-1}$$

defines a cellular decomposition such that $Z_i \setminus Z_{i+1} = Y_i \simeq \mathbf{C}^{n_i}$, hence $A_*(X) \simeq H_{2*}(X)$ and has basis $\{V(\tau_i)\}_i$. □

Corollary 3.1. *If $L \subset N$ has index ℓ, then $A_*(X(F,N)) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell$ is free over $\mathbf{Z}_\ell$. (Equivalently, the torsion elements of $A_*(X(F,N))$ have order dividing some power of ℓ.)*

Proof. Recall that for a toric variety $X = X(F,N)$, there is an exact sequence

$$\prod_{\sigma \in F^{(k-1)}} \sigma^\perp \cap M \to Z^k(X) \to A^k(X) \to 0,$$

and by applying exact functor $-\otimes_{\mathbf{Z}}\mathbf{Z}_\ell$, we get an exact sequence

$$\prod_{\sigma\in F^{(k-1)}}\sigma^\perp\cap M_\ell\to Z^k(X)_\ell\to A^k(X)_\ell\to 0.$$

Since L has finite index ℓ, $M\subset L^*$ has index ℓ too, thus $M_{\langle\ell\rangle}=L^*_\ell$. The later sequence can be read as the localization of

$$\prod_{\sigma\in F^{(k-1)}}\sigma^\perp\cap L^*\to Z^k(X(F,L))_\ell\to A^k(X(F,L))\to 0$$

at ℓ, so $A^k(X(F,N))_\ell\simeq A^k(X(F,L))_\ell$ while $A^k(X(F,L))$ is free by Lemma 3.1. $\square$

Corollary 3.2. *If $x_i,y_j(0\le i\le r,0\le j\le s)$ are coprime to ℓ, then*

1. *$A_*(X(F,N))\hookrightarrow A_*(X(F,N))_{xy}$ where $x=x_0\cdots x_r,\ y=y_0\cdots y_s$;*
2. *there exists $N'\subset N$ of index xy/y_s such that*

   $$L'=N'\cap L=\langle x_0v_0,\dots,x_rv_r,y_0w_0,\dots,y_{s-1}w_{s-1}\rangle,$$

   and the finite toric morphism $X(F,N')\xrightarrow{xy/y_s}X(F,N)$ corresponds to the inclusion $(F,N')\to(F,N)$, and it gives

   $$\begin{array}{ccc} A_*(X(F,N'))_{xy} & \xrightarrow[\sim]{f_*} & A_*(X(F,N))_{xy}\\ \uparrow & & \uparrow\\ A_*(X(F,N')) & \xrightarrow{f_*} & A_*(X(F,N)). \end{array}$$

   In particular, if $L=N$, then $A_*(X(F,N))_{xy}\simeq\mathbf{Z}_{xy}[\xi,h]/((\xi-h)^{r+1}\xi,h^{s+1})$.

Remark. $X(F,N')$ has $x_i=y_j=1(\forall i,j)$ and $[N':L']=\ell$.

Proof.

1. Since x_i,y_j coprime to ℓ, which annihilates the torsions, the localization is exact.
2. If we define $L'=\langle x_0v_0,\dots,x_rv_r,y_0w_0,\dots,y_{s-1}w_{s-1}\rangle$, then $L'\subset L\subset N$ has index $\ell\cdot xy/y_s$ in N. Since ℓ is coprime to xy/y_s, by the structure theorem for finitely generated $\mathbf{Z}$-modules, there exists a sublattice $N'\subset N$ of order xy/y_s, and $N'\cap L=L'$. The finite morphism $X(F,N')\to X(F,N)$ restricts to a finite morphism $V(\sigma_{IJ},N')\twoheadrightarrow V(\sigma_{IJ},N)$ with degree c_{IJ} dividing xy for all $I\subsetneq[r+1]$, $J\subsetneq[s]$. Therefore $f_*:A_*(X(F,N'))\to A_*(X(F,N))$ maps $V(\sigma_{IJ})$ to $c_{IJ}V(\sigma_{IJ}),c_{IJ}|xy$, hence $(f_*)_{xy}$ is an isomorphism.

$\square$

3.2 The result for special case $L = N,\ y_r = 1$.

Let $X = X(F,N),\ X' = X(F',N)$.

Proposition 3.1. *If $y_r = 1$, $v = x_0v_0 + 2x_1v_1 + \cdots + (r+1)x_rv_r + (r+2)y_0w_0 + \cdots + (2r+1)y_{r-1}w_{r-1}$ is a generic element and $\Delta(v) = (\star) \cup (\star\star) \cup (\star\star\star)$.*

Proof. Reduced to the smooth case by positively scaling the basis elements. □

Recall that if σ has primitive generators $e_i,\ i \in I$ then $\prod_{i\in I} D_{e_i} = \frac{1}{\mathrm{mult}(\sigma)} V(\sigma)$. Let $\alpha_i = \gcd(x_i,\ldots,x_r),\ \alpha^i = \gcd(x_0,\ldots,x_i),\ \beta_j = \gcd(y_j,\ldots,y_{r-1})$.
In terms of $h,\ \xi,\ h',\ \xi'$, in the three cases $(\star),\ (\star\star),\ (\star\star\star)$,

$$
\begin{aligned}
(\star) V(\sigma_{IJ}) &= \left(\prod_{i=0}^{i-1} x_i\right)(\xi - h)^i \xi \\
V(\sigma_{I'J'}) &= \left(\prod_{i=i+1}^{r-1} x_i\right)\left(\prod_{j=0}^{r-1} y_j\right) h'^{r-i}(\xi' - h')^r && (0 \le i \le r) \\
(\star\star) V(\sigma_{IJ}) &= \beta_j \left(\prod_{i=0}^{i-1} x_i\right)\left(\prod_{j=0}^{j-1} y_j\right)(\xi - h)^i \xi h^{j+1} \\
V(\sigma_{I'J'}) &= \left(\prod_{i=i+1}^{r} x_i\right)\left(\prod_{j=j+1}^{r-1} y_j\right) h'^{r-i}(\xi' - h')^{r-1-j} && (0 \le i \le r,\ 0 \le j \le r-1) \\
(\star\star\star) V(\sigma_{IJ}) &= \left(\prod_{i=0}^{i-1} x_i\right)(\xi - h)^i \\
V(\sigma_{I'J'}) &= \alpha^i \left(\prod_{i=i+1}^{r-1} x_i\right)\left(\prod_{j=0}^{r-1} y_j\right) h'^{r-i}(\xi' - h')^{r+1} && (0 \le i \le r)
\end{aligned}
$$

respectively. By direct computation,

$$
m_{\sigma_{IJ}\times\sigma_{I'J'}} = [N : N_{\sigma_{IJ}} + N_{\sigma_{I'J'}}] = \begin{cases} x_i & (0 \le i \le r)(\star) \\ x_i y_j/\beta_j & (0 \le i \le r,\ 0 \le j \le r-1)(\star\star) \\ x_i/\alpha^i & (0 \le i \le r)(\star\star\star) \end{cases},
$$

we find that

$$
m_{\sigma_{IJ}\times\sigma_{I'J'}} V(\sigma_{IJ}) \otimes V(\sigma_{I'J'}) = xy \cdot \prod_{i\in I, j\in J} \frac{D_{v_i}}{x_i}\frac{D_{w_j}}{y_j}. \tag{2}
$$

Hence over $\mathbf{Q}$, $[\overline{\Gamma}_f]$ has decomposition

$$
\begin{aligned}
[\overline{\Gamma}_f] = &\sum_{i=0}^{r}\sum_{j=0}^{r} xy \cdot \xi(\xi - h)^i h^j \otimes h'^{r-i}(\xi' - h')^{r-j} \\
&+ \sum_{i=0}^{r} xy(\xi - h)^i \otimes h'^{r-i}(\xi' - h')^{r+1}.
\end{aligned}
$$

Proposition 3.2. $[\overline{\Gamma}_f] : A_*(X)_{\mathbf{Q}} \to A_*(X')_{\mathbf{Q}}$ *is an isomorphism and moreover, for* $0 \leq p \leq r+1,\ 0 \leq q \leq r,$

$$[\overline{\Gamma}_f](\xi - h)^p h^q = \begin{cases} (\xi' - h')^q h'^p & p+q \leq r \\ (\xi' - h')^q h'^p + (-1)^{r-q}(\xi' - h')^{r+1} h'^{p+q-r-1} & p+q \geq r+1 \end{cases}$$

Proof. The proof in section 2 can be adapted here after observing: For $i \geq 0,\ i+j = 2r+1$,

$$\deg(\xi - h)^i h^j = \begin{cases} (-1)^{r-j}\dfrac{1}{xy} & j \leq r \\ 0 & j \geq r+1. \end{cases}$$

□

Theorem 3.1. *If* $r = s,\ x_r = y_r = 1,\ x_{i+1}|x_i,\ y_{j+1}|y_j\ \forall i,j$, *then there exists a correspondence* $\mathscr{F} \in A_*(X \times X')$ *inducing an isomorphism* $A_*(X) \to A_*(X')$ *with inverse* $\mathscr{F}^* \in A_*(X' \times X)$. *In fact,* $\mathscr{F}$ *is exactly the graph closure.*

Proof. Let $\mathscr{F}$ be the graph closure, by previous proposition, $\mathscr{F} : A_*(X)_{\mathbf{Q}} \to A_*(X')_{\mathbf{Q}}$ is an isomorphism with inverse $\mathscr{F}^*$. To show that it induces isomorphism on integral chow group, I claim that $\mathscr{F}(A_*(X)) \subset A_*(X')$ when regarding $A_*(X)$ as subspace of $A_*(X)_{\mathbf{Q}}$. By Lemma 3.1, $A_*(X)$ and $A_*(X')$ have bases

$$\{V(\sigma_{[i,r+1],[j,r]}) : 1 \leq i \leq r+2,\ 1 \leq j \leq r+1\},$$
$$\{V'(\sigma_{[j,r],[i,r+1]}) : 1 \leq i \leq r+2,\ 1 \leq j \leq r+1\}$$

respectively. Over $\mathbf{Q}$,

$$\begin{aligned} V(\sigma_{[i,r+1],[j,r]}) &= m_{ij} \prod_{k=i}^{r-1} x_k \prod_{k=j}^{r-1} y_k (\xi - h)^{r+1-i} \xi h^{r+1-j} \\ &= m_{ij} \prod_{k=i}^{r-1} x_k \prod_{k=j}^{r-1} y_k \left((\xi - h)^{r+2-i} h^{r+1-j} + (\xi - h)^{r+1-i} h^{r+2-j}\right) \end{aligned}$$

where $m_{ij} = \mathrm{mult}(\sigma_{[i,r+1],[j,r]}) = \mathrm{mult}(\sigma_{[i,r],[j,r+1]})$. Then by previous proposition,

$$\begin{aligned} \mathscr{F}(V(\sigma_{[i,r+1],[j,r]})) &= m_{ij} \prod_{k=i}^{r-1} x_k \prod_{k=j}^{r-1} y_k \left((\xi' - h')^{r+1-j} h'^{r+2-j} + (\xi' - h')^{r+2-j} h'^{r+1-i}\right) + \varepsilon \\ &= V'(\sigma_{[i,r],[j,r+1]}) + \varepsilon \end{aligned}$$

where

$$\varepsilon = \delta((m_{ij} \prod_{k=i}^{r-1} x_k \prod_{k=j}^{r-1} y_k \left((-1)^{j-1}(\xi' - h')^{r+1} h^{r+2-i-j} + (-1)^{j-2}(\xi' - h')^{r+1} h^{r+2-i-j}\right))) = 0,$$

for $i \neq r+2,\ j \neq 1$. For $i = r+2$, the same formula holds. For $i = r+2$, $V(\sigma_{[i,r],[j,r+1]}) = m_{r+2,j} \prod_{k=j}^{r-1} y_k h^{r+1-j}$,

$$\mathscr{F}(V(\sigma_{[i,r],[j,r+1]})) = \mathrm{mult}(\langle w_j, \ldots, w_r \rangle) \prod_{k=j}^{r-1} y_k (\xi' - h')^{r+1-j} = V(\langle w_j, \ldots, w_r \rangle).$$

For $i \le r+1$, $j=1$, $V(\sigma_{[i,r],[1,r+1]}) = m_{i1}\prod_{k\ge i} x_k \prod_{k\ge 1} y_k (\xi-h)^{r+2-j}h^r$,

$$\begin{aligned}
\mathscr{F}(V(\sigma_{[i,r],[j,r+1]})) &= m_{i1}\prod_{k\ge i} x_k \prod_{k\ge 1} y_k (\xi'-h')^r h'^{r+2-i} \\
&= \frac{\mathrm{mult}(\langle v_i,\dots,v_{r+1},w_1,\dots,w_r\rangle)}{\mathrm{mult}(v_{i-1},\dots,v_r,w_1,\dots,w_r)x_{i-1}} V(\langle v_{i-1},\cdots,v_r,w_1,\dots,w_r\rangle) \\
&= \frac{y_0\cdot \gcd(x_0,\dots,x_{i-1})}{\gcd(x_0,\dots,x_{i-2},y_0)\cdot x_{i-1}} V(\langle v_{i-1},\cdots,v_r,w_1,\dots,w_r\rangle) \\
&= \frac{y_0}{\gcd(x_0,\dots,x_{i-2},y_0)} V(\langle v_{i-1},\cdots,v_r,w_1,\dots,w_r\rangle) \in A_*(X').
\end{aligned}$$

□

3.3 The result for more general case $L \subset N$, $x_r = y_r = 1$

Theorem 3.2. *The graph closure $[\overline{\Gamma}_f] : A_*(X)_{\mathbf{Q}} \to A_*(X')_{\mathbf{Q}}$ induces an isomorphism, the inverse is $[\overline{\Gamma}_{f^{-1}}]$.*

Proof. In this case $A_*(X) \simeq \mathbf{Q}[\xi,h]/((\xi-h)^{r+1}\xi,h)$,

$$D_{v_{r+1}} \mapsto \xi,\ D_{w_i} \mapsto y_i h,\ D_{v_i} \mapsto x_i(\xi-h),\ 0\le i\le r.$$

Note that $m_{\sigma_{IJ}\times\sigma_{I'J'}} V(\sigma_{IJ})\otimes V(\sigma_{I'J'}) = [N:L]xy\cdot \prod_{i\in I, j\in J} \frac{D_{v_i}}{x_i}\frac{D_{w_j}}{y_j}$ Indeed, we have seen that this is true when $L=N$, (Equation 2). For general $L\subset N$,

$$\begin{aligned}
& m_{\sigma_{IJ}\times\sigma_{I'J'}}\mathrm{mult}_N(\sigma_{IJ})\mathrm{mult}_N(\sigma_{I'J'}) \\
&= [N : N_{\sigma_{IJ}} + N_{\sigma_{I'J'}}][N_{\sigma_{IJ}} : \langle v_i, w_j : i\in I, j\in J\rangle_{\mathbf{Z}}][N_{\sigma_{I'J'}} : \langle v_i, w_j : i\in I', j\in J'\rangle_{\mathbf{Z}}] \\
&= [N : N_{\sigma_{IJ}} + N_{\sigma_{I'J'}}][N_{\sigma_{IJ}} + N_{\sigma_{I'J'}} : \langle v_i, w_j : i\in I\cup I', j\in J\cup J'\rangle_{\mathbf{Z}}] \\
&= [N : \langle v_i, w_j : i\in I\cup I', j\in J\cup J'\rangle_{\mathbf{Z}}] \\
&= [N:L][L : \langle v_i, w_j : i\in I\cup I', j\in J\cup J'\rangle_{\mathbf{Z}}] \\
&= [N:L][L : L_{\sigma_{IJ}} + L_{\sigma_{I'J'}}]\mathrm{mult}_L(\sigma_{IJ})\mathrm{mult}_L(\sigma_{I'J'}) \\
&= [N:L] m^L_{\sigma_{IJ}\times\sigma_{I'J'}}\mathrm{mult}_L(\sigma_{IJ})\mathrm{mult}_L(\sigma_{I'J'}).
\end{aligned}$$

By Equation 2 for $L=N$,

$$\begin{aligned}
m_{\sigma_{IJ}\times\sigma_{I'J'}} V(\sigma_{IJ})\otimes V(\sigma_{I'J'}) &= [N:L] m^L_{\sigma_{IJ}\times\sigma_{I'J'}}\mathrm{mult}_L(\sigma_{IJ})\mathrm{mult}_L(\sigma_{I'J'}) V(\sigma_{IJ})\otimes V(\sigma_{I'J'}) \\
&= [N:L]xy\cdot \prod_{i\in I, j\in J} \frac{D_{v_i}}{x_i}\frac{D_{w_j}}{y_j}.
\end{aligned}$$

Hence over $\mathbf{Q}$, $[\overline{\Gamma}_f]$ has decomposition

$$[\overline{\Gamma}_f] = [N:L]\cdot xy\left(\sum_{i=0}^r\sum_{j=0}^r \xi(\xi-h)^i h^j \otimes h'^{r-i}(\xi'-h')^{r-j} + \sum_{i=0}^r (\xi-h)^i \otimes h'^{r-i}(\xi'-h')^{r+1}\right).$$

By noticing that for $i+j=2r+1$,

$$\deg(\xi-h)^i h^j = \begin{cases} (-1)^{r-j}\dfrac{1}{[N:L]xy} & j \le r \\ 0 & j \ge r+1. \end{cases},$$

we can conclude that $[\overline{\Gamma}_f]$ is an isomorphism using similar calculation as in the smooth case (Corollary 2.2). $\square$

4 Examples

4.1 $r=1,\ L=N$

In order to interpret our main result in modern language, we would like to give a simplest nontrivial example in this section. Let $r=1,\ L=\langle v_0,v_1,w_0\rangle = N \simeq \mathbf{Z}^3,\ -v_2 = xv_0+v_1 = xw_0+w_1$. We get

$$A_0(X)=\mathbf{Z}[pt],\ A_1(X)=\mathbf{Z}[V(\sigma_{2,1})]\oplus\mathbf{Z}[V(\sigma_{[1,2],\emptyset}],$$

$$A_2(X)=\mathbf{Z}[V(\sigma_{\emptyset,1})]\oplus\mathbf{Z}[V(\sigma_{2,\emptyset})],\ A_3(X)=\mathbf{Z}[X]$$

$$A_0(X')=\mathbf{Z}[pt],\ A_1(X')=\mathbf{Z}[V'(\sigma_{1,2})]\oplus\mathbf{Z}[V'(\sigma_{\emptyset,[1,2]})],$$

$$A_2(X')=\mathbf{Z}[V'(\sigma_{1,\emptyset})]\oplus\mathbf{Z}[V'(\sigma_{\emptyset,2})],\ A_3(X')=\mathbf{Z}[X'],$$

$$\xi=V(e_5)=V(\sigma_{2,\emptyset}), h=V(e_4)=xV(e_3), V(e_1)=x(\xi-h), V(e_2)=(\xi-h).$$

Also we write down all components appearing in the proof and determine the graph correspondence.

$$\begin{aligned}(\star)A &= V(\sigma_{2,\emptyset})\otimes V(\sigma_{1,0}), m=x\\ B &= V(\sigma_{\{0,2\},\emptyset})\otimes V(\sigma_{\emptyset,0}), m=1\\ (\star\star)C &= V(\sigma_{2,1})\otimes V(\sigma_{1,\emptyset}), m=x^2\\ D &= V(\sigma_{\{0,2\},1})\otimes[X'], m=x\\ (\star\star\star)E &= [X]\otimes V(\sigma_{1,\{0,1\}}), m=x\\ F &= V(\sigma_{0,\emptyset})\otimes V(\sigma_{\emptyset,\{0,1\}}), m=1\end{aligned}$$

$$[\overline{\Gamma}_f]=xA+B+x^2C+xD+xE+F$$

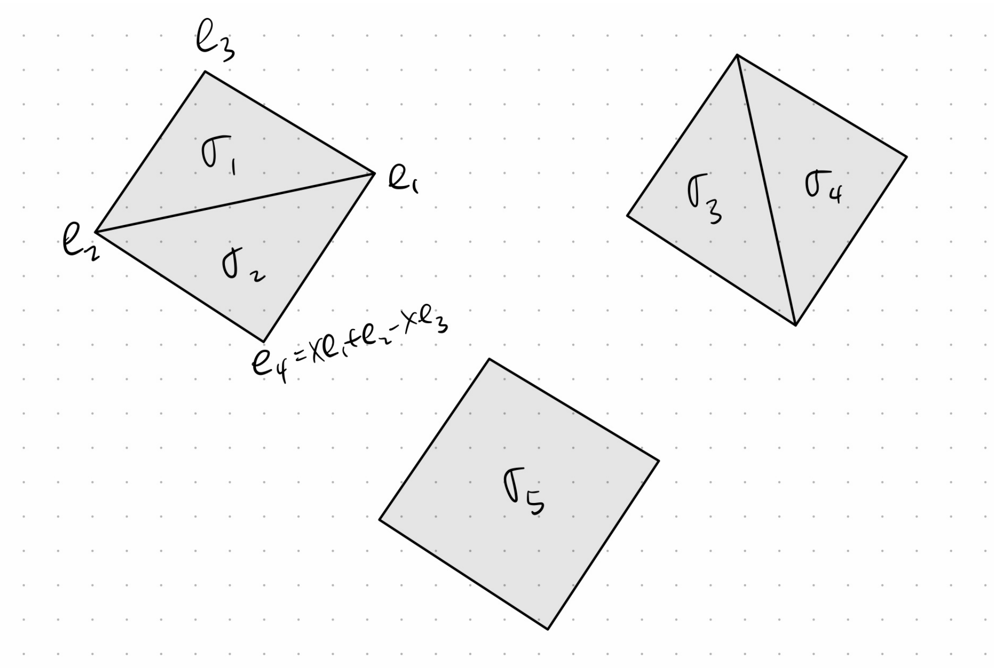
In fact, we also expect that the graph closure should coincide with the fiber product and thus the fiber product identify the chow motives. However, in general the fiber product is even difficult to be determined. Fortunately, in this simple case, we can prove it.

Proposition 4.1. *The fiber product $X\times_{X_0}X'$ is a toric variety with fan $F\times_{F_0}F'$. (Here $F\times_{F_0}F'$ is defined to be $\{\sigma\cap\sigma' : \exists\sigma_0\in F_0 \text{ s.t. } \sigma\cap\sigma'\subset\sigma_0\}$).*

Proof. The functoriality of $\tau \mapsto U_\tau$ and the universal property of fiber product in the category of schemes induces a commutative diagram

$$\begin{array}{ccccc} & & U_{\sigma\cap\sigma'} & & \\ & \swarrow & \downarrow \exists!\varphi & \searrow & \\ U_\sigma & \longleftarrow & U_\sigma \times_{U_{\sigma_0}} U_{\sigma'} & \longrightarrow & U_{\sigma'} \\ & \searrow & \downarrow & \swarrow & \\ & & U_{\sigma_0} & & \end{array}$$

We only have to take care of the exceptional part. Write e_1, e_2, e_3, e_4, e_5 for v_0, v_1, w_0, w_1, v_2 respectively. The toric flop can be illustrated as:

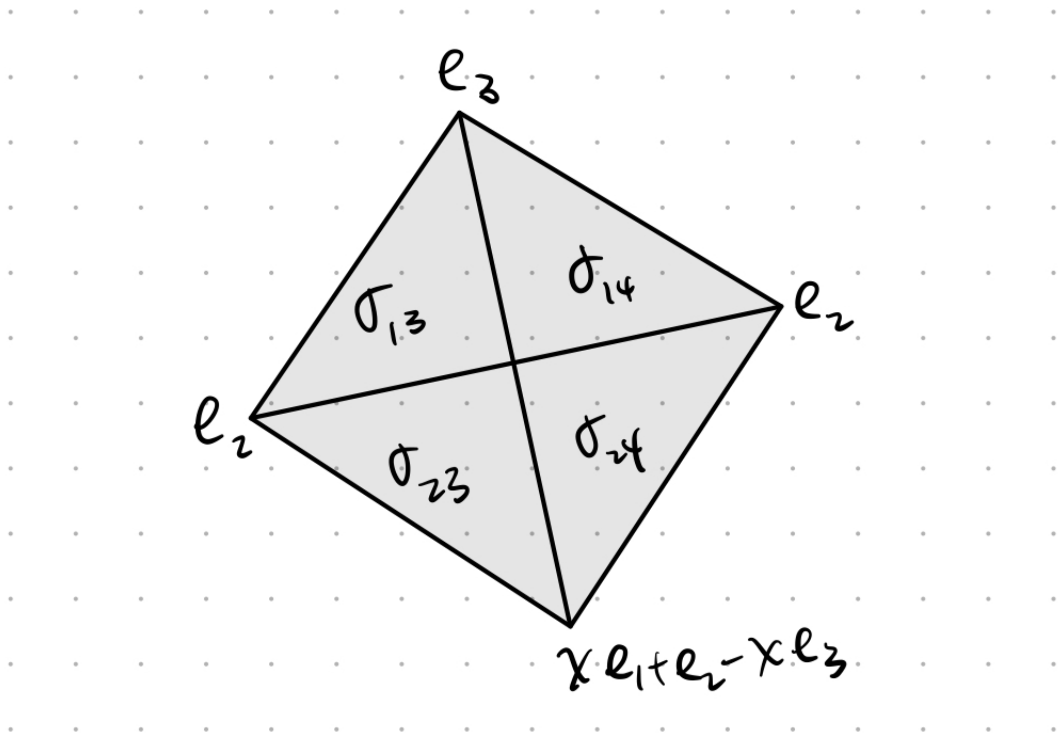


I claim that φ is an isomorphism by showing that it induces isomorphism on coordinate rings. A direct computation shows that

$$\begin{aligned}
\sigma_1^\vee \cap M &= \langle e_1, e_2, e_3 \rangle \\
\sigma_2^\vee \cap M &= \langle e_1 + e_3, xe_2 + e_3, -e_3, e_2 \rangle \\
\sigma_3^\vee \cap M &= \langle xe_2 + e_3, e_1 - xe_2, e_2 \rangle \\
\sigma_4^\vee \cap M &= \langle e_1 + e_3, -e_1 + xe_2, e_1, e_2 \rangle \\
\sigma_5^\vee \cap M &= \langle e_1, e_2, xe_2 + e_3, e_1 + e_3, e_1 + e_2 + e_3 \rangle.
\end{aligned}$$

$$
\begin{aligned}
A_{\sigma_1} \otimes_{A_{\sigma_5}} A_{\sigma_3} &= \mathbf{C}[u,v,w] \otimes_{\mathbf{C}[w,v,uw,v^x w,uvw]} \mathbf{C}[v^x w, uv^{-x}, v] \\
&= \mathbf{C}[1 \otimes uv^{-x}, v \otimes 1, w \otimes 1] \\
A_{\sigma_1} \otimes_{A_{\sigma_5}} A_{\sigma_4} &= \mathbf{C}[u,v,v] \otimes_{\mathbf{C}[u,v,uw,v^x w,uvw]} \mathbf{C}[uw, u^{-1}v^x, u, v] \\
&= \mathbf{C}[u \otimes 1, v \otimes 1, 1 \otimes u^{-1}v^x, w \otimes 1] \\
&\simeq \mathbf{C}[U,V,W,T]/(VW - T^x) \\
A_{\sigma_2} \otimes_{A_{\sigma_5}} A_{\sigma_3} &= \mathbf{C}[v^x w, uw, w^{-1}, v] \otimes_{\mathbf{C}[u,v,uw,v^x w,uvw]} \mathbf{C}[v^x w, uv^{-x}, v] \\
&= \mathbf{C}[1 \otimes uv^{-x}, v \otimes 1, w^{-1} \otimes 1, v^x w \otimes 1] \\
&\simeq \mathbf{C}[U,V,W,T]/(TV - U^x) \\
A_{\sigma_2} \otimes_{A_{\sigma_5}} A_{\sigma_4} &= \mathbf{C}[v^x w, uw, w^{-1}] \otimes_{\mathbf{C}[u,v,uw,v^x w,uvw]} \mathbf{C}[uw, u^{-1}v^x, u, v] \\
&= \mathbf{C}[uw \otimes 1, w^{-1} \otimes 1, 1 \otimes u^{-1}v^x, v \otimes 1] \\
&\simeq \mathbf{C}[U,V,W,T]/(W^x - UVT)
\end{aligned}
$$

Over the exceptional set $V(\sigma_0)$, $F \times_{F_0} F'$ is the simplicial subdivision of σ_0 by adding $v_0 = xe_1 + e_2$.



A direct computation shows that

$$
\begin{aligned}
\sigma_{13}^{\vee} \cap M &= \langle e_1 - xe_2, e_2, e_3 \rangle \\
\sigma_{14}^{\vee} \cap M &= \langle e_1, -e_1 + xe_2, e_1, e_2 \rangle \\
\sigma_{23}^{\vee} \cap M &= \langle xe_2 + e_3, e_1 - xe_2, -e_3, e_2 \rangle \\
\sigma_{24}^{\vee} \cap M &= \langle e_1 + e_3, -e_1 + xe_2, -e_3, e_2 \rangle.
\end{aligned}
$$

$$A_{\sigma_{13}} = \mathbf{C}[uv^{-x}, v, w]$$
$$A_{\sigma_{14}} = \mathbf{C}[u, v, u^{-1}v^x, w]$$
$$A_{\sigma_{23}} = \mathbf{C}[uv^{-x}, v, w^{-1}, v^x w]$$
$$A_{\sigma_{24}} = \mathbf{C}[uw, w^{-1}, u^{-1}v^x, v]$$

By identifying $u \otimes 1, v \otimes 1, w \otimes 1$, with u, v, w, we deduce that $U_{\sigma_i} \times_{U_{\sigma_0}} U_{\sigma_j} \simeq U_{\sigma_{ij}}$ for $i \neq j$.

4.2 $r = 1,\ L \subsetneq N$

This example considers nonzero coprime integers m, n with m odd. L a sublattice of $N = \mathbf{Z}^3$ generated by $v_0 = (m, n, 0)$, $v_1 = (0, n, 1)$, $w_0 = (0, 0, 1)$, the index is mn. Set $x_i = y_j = 1\ \forall i, j$. First note that in general, for $\mathscr{F} \in A^*(X \times X')$, $\mathscr{F}(x) = 0$ for $x \in A^*(X)_{\text{tor}}$ since the intersection numbers take value in $\mathbf{Q}$, which is torsion free. By Corollary 3.2, this covers the cases with x_i, y_i coprime with mn. We have seen that the graph closure identifies rational Chow groups, in this case we show that it identifies free parts of integral Chow groups. By direct computations, the Chow groups are

$$A_3(X) = \mathbf{Z}[X]$$
$$A_2(X) \simeq \mathbf{Z}[V(w_1)] \oplus \mathbf{Z}[V(v_2)] \oplus A_2(X)_{\text{tor}}$$
$$A_1(X) \simeq \mathbf{Z}[V(v_1v_2)] \oplus \mathbf{Z}[V(v_2w_1)] \oplus A_1(X)_{\text{tor}}$$
$$A_0(X) \simeq \mathbf{Z}[pt]$$

$$A_3(X') = \mathbf{Z}[X']$$
$$A_2(X') \simeq \mathbf{Z}[V'(v_1)] \oplus \mathbf{Z}[V'(v_2)] \oplus A_2(X')_{\text{tor}}$$
$$A_1(X') \simeq \mathbf{Z}[V'(v_1v_2)] \oplus \mathbf{Z}[V'(v_2w_1)] \oplus A_1(X')_{\text{tor}}$$
$$A_0(X') \simeq \mathbf{Z}[pt]$$

where

$$A_2(X)_{\text{tor}} \simeq A_2(X') \simeq \mathbf{Z}/mn\mathbf{Z}$$
$$A_1(X)_{\text{tor}} \simeq A_1(X') \simeq \mathbf{Z}/m \oplus \mathbf{Z}/mn\mathbf{Z}.$$

The graph closures decompose into

$$[\overline{\Gamma}_f] = mA + mnB + mnC + D + E + mnF)$$
$$[\overline{\Gamma}_f]^* = [\overline{\Gamma}_{f^{-1}}] = mA^* + mnB^* + mnC^* + D^* + E^* + mnF^*$$

where

$$A = V(v_2) \otimes V'(v_1w_0) \qquad A^* = V'(v_2) \otimes V(w_1v_0)$$
$$B = V(v_0v_2) \otimes V'(w_0) \qquad B^* = V'(w_0v_2) \otimes V(v_0)$$
$$C = V(v_2w_1) \otimes V'(v_1) \qquad C^* = V'(v_2v_1) \otimes V(w_1)$$
$$D = pt \otimes X' \qquad D^* = pt \otimes X$$
$$E = X \otimes pt \qquad E^* = X' \otimes pt$$
$$F = V(v_0) \otimes V'(w_0w_1) \qquad F^* = V'(w_0) \otimes V(v_0v_1)$$

Let $\xi = V(v_2)$, $h = V(w_0)$, then $V(v_0) = V(v_1) = \xi - h$, $V(w_1) = h$ and $A^*(X) = \mathbf{Q}[\xi,h]/(\xi(\xi-h)^2,h^2)$. Over $\mathbf{Q}$,

$$\begin{aligned}[\overline{\Gamma}_f] = mn(&\xi \otimes h'(\xi'-h') + (\xi-h)\xi \otimes (\xi-h) + (\xi-h)\xi \otimes h \\ &+ (\xi-h)^2 h \otimes 1 + 1 \otimes (\xi'-h')^2 h' + (\xi-h) \otimes (\xi'-h')^2).\end{aligned}$$

$[\overline{\Gamma}_f]$ preserves free parts, hence induces isomorphism with inverse $[\overline{\Gamma}_f]^*$. To see this, note that

$$\begin{aligned}
[\overline{\Gamma}_f]([X]) &= [X'] \\
[\overline{\Gamma}_f]([V(w_1)]) &= [V'(w_0)] = [V'(v_2)] - [V'(v_1)] \\
[\overline{\Gamma}_f]([V(v_2)]) &= [V'(w_0)] + [V'(v_1)] = [V'(v_2)] \\
[\overline{\Gamma}_f]([V(v_1v_2)]) &= \frac{1}{n}[V'(v_1w_0)] = [V(v_0v_1)] + 2[V(v_1w_1)] - [V(v_1v_2)] \\
[\overline{\Gamma}_f]([V(v_2w_1)]) &= \frac{1}{n}[V'(v_1w_0)] + [V'(w_0w_1)] = [V(v_0v_1)] + 2[V(v_1w_1)] - [V(v_1v_2)] + [V'(v_0w_1)] \\
[\overline{\Gamma}_f](pt) &= pt,
\end{aligned}$$

the coefficients are integral, i.e.

$$[\overline{\Gamma}_f](A^*(X)_{\text{free}}) \subset A^*(X')_{\text{free}},$$

regarding $A^*(X)_{\text{free}} = A^*(X)/A^*(X)_{\text{tor}}$ as a subgroup of $A^*(X)_{\mathbf{Q}}$. Similarly,

$$[\overline{\Gamma}_f]^*(A^*(X')_{\text{free}}) \subset A^*(X)_{\text{free}},$$

hence $[\overline{\Gamma}_f]$ induces an isomorphism $A^*(X)_{\text{free}} \xrightarrow{\sim} A^*(X')_{\text{free}}$. □

References

[1] Willian Fulton. *Introduction to Toric Varieties*. Princeton University Press.

[2] Willian Fulton and Bernd Sturmfels. *Intersection Theory on Toric Varieties*. arXiv: alg-geom/9403002.

[3] Yuan-Pin Lee, Hui-Wen Lin, and Chin-Lung Wang. “Flops, motives and invariance of quantum rings”. In: *Annals of Mathematics* 172.1 (2010), pp. 243–290.

[4] Yuan-Pin Lee, Hui-Wen Lin, and Chin-Lung Wang. “Towards A + B theory in conifold transitions for Calabi-Yau threefolds”. In: *Journal of Differential Geometry* 110.3 (2018), pp. 495–541.

[5] Miles Reid. “Decomposition of Toric Morphisms”. In: *Arithmetic and Geometry* (1983), pp. 395–418.